Charge and current density profiles of a degenerate magnetized free-electron gas near a hard wall

1998 J. Phys. A: Math. Gen. 31 6547

(http://iopscience.iop.org/0305-4470/31/31/002)

J. Phys. A: Math. Gen. **31** (1998) 6547–6560. Printed in the UK PII: S0305-4470(98)89454-7

Charge and current density profiles of a degenerate magnetized free-electron gas near a hard wall

M M Kettenis and L G Suttorp
Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

Received 24 November 1997

Abstract. The charge and current densities of a completely degenerate free-electron gas in a uniform magnetic field are found to have a damped oscillatory spatial dependence near a wall that is parallel to the magnetic field. For large distances from the wall the behaviour of the associated profile functions is analysed by means of systematic asymptotic expansions. Both densities are shown to decay to their bulk values through a Gaussian tail, with prefactors that depend algebraically and logarithmically on the distance from the wall.

1. Introduction

Ever since Landau's original derivation [1] of diamagnetism in a magnetized free-electron gas, there has been interest in boundary effects. This is not surprising, since the diamagnetism of a finite sample is caused by currents flowing near the boundary. To gain a deeper insight into the diamagnetic effect one needs to investigate the behaviour of these currents in the neighbourhood of a wall parallel to the external magnetic field. Of particular interest is the question of how the current density decays in the bulk.

For high temperatures it is adequate to use Maxwell–Boltzmann statistics. In that approximation the precise form of the current profile in the neighbourhood of a hard wall has been studied by Ohtaka and Moriya [2] and by Jancovici [3] within the framework of linear response and, more recently, by John and Suttorp [4] with the use of a Green function method. In both approaches a Gaussian decay of the current density in the bulk has been found: the decay is proportional to $\exp(-x^2)$, with x the distance from the wall in suitable units. A similar decay has been found [4, 5] for the excess charge density and the excess (kinetic) pressure.

For lower temperatures the effects of quantum statistics have to be taken into account. In that regime Macris *et al* [6] derived an exponential bound ($\sim\exp(-x)$) on the decay of the current density in the bulk, at least for non-zero temperature. For the strongly degenerate case of vanishing T, Ohtaka and Moriya [2] and Jancovici [3] obtained a closed expression for the current density, via an inverse Laplace transform of the expression for Maxwell–Boltzmann statistics. Remarkably enough, their results exhibit a much slower algebraic decay proportional to x^{-1}. Using the same method one easily derives similar expressions for the excess charge density and the excess pressure at $T = 0$. However, the expression for the excess pressure obtained along these lines shows the unphysical feature of an oscillatory behaviour that is no longer damped in the bulk.

The various findings for the asymptotic behaviour of physical quantities near the bulk, as described above, justify a closer inspection of the problem. In this paper we will derive systematic asymptotic expansions for the charge and the current density near the bulk by starting from exact integral expressions valid at $T = 0$, which will be established on the basis of a Green function formulation. The validity of these asymptotic expansions will be assessed by a comparison with the results of a numerical evaluation of the integral expressions.

2. Green functions; charge and current density

Consider the half-space $x > 0$, with a hard wall at $x = 0$. Choose the magnetic field in the z-direction, with vector potential $\boldsymbol{A} = (0, Bx, 0)$. The transverse part of the Hamiltonian for a particle with charge e and mass m in this field is given by

$$H_\perp = -\frac{\hbar^2}{2m}\Delta_\perp + \mathrm{i}\hbar\omega_c x\frac{\partial}{\partial y} + \frac{1}{2}m{\omega_c}^2 x^2 \tag{1}$$

where $\omega_c = eB/mc$ is the cyclotron frequency associated with the particle. The Green function for the eigenvalue equation $H_\perp\psi_n(\boldsymbol{r}) = E_n\psi_n(\boldsymbol{r})$ ($\boldsymbol{r} = (x, y)$) is defined by

$$(H_\perp - z)G_z(\boldsymbol{r}, \boldsymbol{r}') = -\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{2}$$

with z a complex energy variable and with boundary condition $G_z(\boldsymbol{r}, \boldsymbol{r}') = 0$ for $x = 0$ and/or $x' = 0$. This means we can express the Green function as

$$G_z(\boldsymbol{r}, \boldsymbol{r}') = \sum_n \psi_n(\boldsymbol{r})\psi_n^*(\boldsymbol{r}')\frac{1}{z - E_n}. \tag{3}$$

The discontinuity of G_z at $z = E$

$$G_E(\boldsymbol{r}, \boldsymbol{r}') = \frac{\mathrm{i}}{2\pi}[G_{z=E+\mathrm{i}0}(\boldsymbol{r}, \boldsymbol{r}') - G_{z=E-\mathrm{i}0}(\boldsymbol{r}, \boldsymbol{r}')] \tag{4}$$

will be referred to as the energy Green function.

Due to the translation invariance in the y-direction of both the Hamiltonian and the boundary condition, a Fourier transform is appropriate. If we define the transform by $G_z(\boldsymbol{r}, \boldsymbol{r}') = (2\pi)^{-1}\int_{-\infty}^{\infty} \mathrm{d}k\, \exp[\mathrm{i}k(y - y')]G_z(k, x, x')$, the Hamiltonian becomes

$$H_\perp(k) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{\hbar^2}{2m}k^2 - \hbar\omega_c kx + \frac{1}{2}m\omega_c^2 x^2 \tag{5}$$

and the equivalent of (2) is

$$[H_\perp(k) - z]G_z(k, x, x') = -\delta(x - x'). \tag{6}$$

This means that we can write the Fourier transform of the energy Green function as

$$G_E(k, x, x') = \sum_n \psi_n(k, x)\psi_n^*(k, x)\delta(E_n(k) - E) \tag{7}$$

where the $\psi_n(k, x)$ are eigenfunctions of $H_\perp(k)$, with eigenvalues $E_n(k)$, normalized such that $\int_0^\infty \mathrm{d}x\, |\psi_n(k, x)|^2 = 1$.

The charge density for a gas of spin-$\frac{1}{2}$ fermions without mutual interaction at temperature $T = 0$ and chemical potential μ is given by

$$\rho(x) = e\frac{2^{3/2}m^{1/2}}{\pi\hbar}\sum_n |\psi_n(\boldsymbol{r})|^2(\mu - E_n)^{1/2} \tag{8}$$

where the spin degeneracy has been taken into account. With the help of the Fourier transform of the energy Green function (7) we can write this as

$$\rho(x) = e\frac{2^{1/2}m^{1/2}}{\pi^2\hbar}\int_0^\mu \mathrm{d}E\,(\mu - E)^{1/2}\int_{-\infty}^{\infty}\mathrm{d}k\,G_E(k, x, x). \tag{9}$$

Likewise, the current density in the y-direction is given by

$$j_y(x) = e\frac{2^{3/2}m^{1/2}}{\pi\hbar}\sum_n \left\{-\frac{\mathrm{i}\hbar}{2m}\left[\psi_n^*(\boldsymbol{r})\frac{\partial\psi_n(\boldsymbol{r})}{\partial y} - \frac{\partial\psi_n^*(\boldsymbol{r})}{\partial y}\psi_n(\boldsymbol{r})\right] - \omega_c x|\psi_n(\boldsymbol{r})|^2\right\} \times(\mu - E_n)^{1/2} \tag{10}$$

or

$$j_y(x) = e\frac{2^{1/2}m^{1/2}}{\pi^2\hbar}\int_0^\mu \mathrm{d}E\,(\mu - E)^{1/2}\int_{-\infty}^{\infty}\mathrm{d}k\,\left(\frac{\hbar}{m}k - \omega_c x\right)G_E(k, x, x). \tag{11}$$

3. Explicit form of Green function; parabolic cylinder functions

We now define dimensionless quantities by expressing the position x in units $(\hbar/m\omega_c)^{1/2}$ and the wavenumber k in units $(m\omega_c/\hbar)^{1/2}$. We also express all energies in units $\hbar\omega_c$, and scale the Green function accordingly. In this way, we get the following dimensionless Hamiltonian

$$H_\perp(k) = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}(x - k)^2. \tag{12}$$

The corresponding eigenfunctions are the parabolic cylinder functions [7]

$$\psi_n(k, x) = \left[\int_0^\infty \mathrm{d}t\,D_{z_n(k)-1/2}(\sqrt{2}(t - k))\right]^{-1/2} D_{z_n(k)-1/2}(\sqrt{2}(x - k)) \tag{13}$$

where we have applied a similar normalization as before (but now with dimensionless x and k). The function $z_n(k)$, which gives the eigenvalues, is defined by the boundary condition at $x = 0$

$$D_{z_n(k)-1/2}(-\sqrt{2}k) = 0. \tag{14}$$

The function is plotted in figure 1. It has been studied before by MacDonald and Středa [8] and Kunz [9]. As can be seen in figure 1, $z_n(k)$ has the property that $\lim_{k\to\infty} z_n(k) = n + \frac{1}{2}$ and $z_n(0) = 2n + \frac{3}{2}$.

If we substitute (13) into (7) we get the following expression for the (dimensionless) energy Green function

$$G_E(k, x, x) = \sum_n \left[\int_0^\infty \mathrm{d}t\,D^2_{z_n(k)-1/2}(\sqrt{2}(t - k))\right]^{-1} D^2_{z_n(k)-1/2}(\sqrt{2}(x - k))\delta(E - z_n(k)). \tag{15}$$

Inserting this expression in the dimensionless equivalent of (9) and (11) we can carry out the integration over E. Defining $k_n(\mu)$ by $z_n(k_n(\mu)) = \mu$ we arrive at the following expressions for the charge density

$$\rho(x) = \frac{e}{2\pi^2}\left(\frac{2m\omega_c}{\hbar}\right)^{3/2}\sum_n{}' \int_{k_n(\mu)}^\infty \mathrm{d}k\,[\mu - z_n(k)]^{1/2} \times\left[\int_0^\infty \mathrm{d}t\,D^2_{z_n(k)-1/2}(\sqrt{2}(t - k))\right]^{-1} D^2_{z_n(k)-1/2}(\sqrt{2}(x - k)) \tag{16}$$

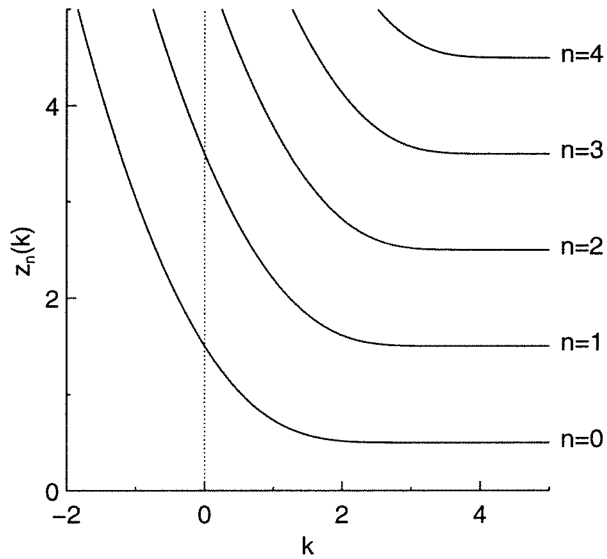

Figure 1. The function $z_n(k)$, for $n = 0$–4.

and the current density

$$j_y(x) = -\frac{e}{2\pi^2}\left(\frac{2m\omega_c}{\hbar}\right)^{3/2}\left(\frac{\hbar\omega_c}{m}\right)^{1/2}\sum_n{}' \int_{k_n(\mu)}^{\infty} \mathrm{d}k\,[\mu - z_n(k)]^{1/2}(x-k) \times\left[\int_0^\infty \mathrm{d}t\, D^2_{z_n(k)-1/2}(\sqrt{2}(t-k))\right]^{-1} D^2_{z_n(k)-1/2}(\sqrt{2}(x-k)). \tag{17}$$

The summations are over all $n < \mu - \frac{1}{2}$, as indicated by the prime. From these expressions it is fairly easy to see that in the bulk the charge density is given by

$$\rho = \lim_{x\to\infty}\rho(x) = \frac{e}{2\pi^2}\left(\frac{2m\omega_c}{\hbar}\right)^{3/2}\sum_n{}'[\mu - (n+\tfrac{1}{2})]^{1/2} \tag{18}$$

and, because of the orthogonality of $D_n(\sqrt{2}k)$ and $kD_n(\sqrt{2}k)$, that there is no current in the bulk.

Alternative expressions for the charge and the current density are found by writing $G_z(k, x, x')$ as the sum of the Green function for an infinite domain and a correction due to the boundaries. The infinite-domain Green function is given by [10]

$$G^0_z(k, x, x') = -\frac{1}{\sqrt{\pi}}\Gamma\left(-z+\tfrac{1}{2}\right) D_{z-1/2}(\sqrt{2}(x-k))D_{z-1/2}(-\sqrt{2}(x'-k)) \tag{19}$$

for $x > x'$, and an analogous expression for $x < x'$. The correction for the chosen geometry is [4]

$$G^c_z(k, x, x') = \frac{1}{\sqrt{\pi}}\Gamma(-z+\tfrac{1}{2})\frac{D_{z-1/2}(\sqrt{2}k)}{D_{z-1/2}(-\sqrt{2}k)}D_{z-1/2}(\sqrt{2}(x-k))D_{z-1/2}(\sqrt{2}(x'-k)) \tag{20}$$

for all $x > 0$ and $x' > 0$.

The energy Green function is determined by the poles of $G^0_z + G^c_z$. The contributions from the gamma functions in G^0 and G^c cancel, so that only the roots of the denominator in

(20) contribute. They give a residue proportional to $[\partial D_{z-1/2}(-\sqrt{2}k)/\partial z]^{-1}$ in $z = z_n(k)$, which results in

$$G_E(k,x,x) = \frac{1}{\sqrt{\pi}} \sum_n \Gamma(-z_n(k) + \tfrac{1}{2}) D^2_{z_n(k)-1/2}(\sqrt{2}(x-k)) \times D_{z_n(k)-1/2}(\sqrt{2}k) \left[\left. \frac{\partial D_{z-1/2}(-\sqrt{2}k)}{\partial z} \right|_{z=z_n(k)} \right]^{-1} \delta(E - z_n(k)). \tag{21}$$

From (14) we see that

$$\left. \frac{\partial D_{z-1/2}(-\sqrt{2}k)}{\partial z} \right|_{z=z_n(k)} = - \left. \frac{\partial D_{z-1/2}(-\sqrt{2}k)}{\partial k} \right|_{z=z_n(k)} \left[\frac{\mathrm{d}z_n(k)}{\mathrm{d}k} \right]^{-1}. \tag{22}$$

With the help of the Wronskian $W[D_\nu(z), D_\nu(-z)] = \sqrt{2\pi}/\Gamma(-\nu)$ [7] we derive

$$G_E(k,x,x) = -\frac{1}{2\pi} \sum_n \Gamma^2(-z_n(k) + \tfrac{1}{2}) D^2_{z_n(k)-1/2}(\sqrt{2}k) D^2_{z_n(k)-1/2}(\sqrt{2}(x-k)) \times \frac{\mathrm{d}z_n(k)}{\mathrm{d}k} \delta(E - z_n(k)). \tag{23}$$

By comparing this with (15) we find

$$\left[\int_0^\infty \mathrm{d}t \, D^2_{z_n(k)-1/2}(\sqrt{2}(t-k)) \right]^{-1} = \frac{1}{2\pi} \Gamma^2(-z_n + \tfrac{1}{2}) D^2_{z_n(k)-1/2}(\sqrt{2}k) \left| \frac{\mathrm{d}z_n(k)}{\mathrm{d}k} \right| \tag{24}$$

where we made use of the fact that $\mathrm{d}z_n(k)/\mathrm{d}k < 0$.

Plugging (24) into (16) and (17) gives alternative expressions for the charge density and the current density. Unfortunately neither these nor (16) and (17) allow us to evaluate the integrals over k analytically. Both sets of formulae can be used for a numerical evaluation, although the expressions based on (23) are more convenient, since they involve a single integration only. Numerical results obtained along these lines are presented in figure 2. Both the charge and current density decay to their bulk values within a distance of a few times the typical lengthscale of the system $(\hbar/m\omega_c)^{1/2}$. Near the boundary, the current density exhibits a layered structure of currents flowing in alternate directions. The number of layers increases with the number of filled Landau levels.

4. Asymptotic expansions

Expressions (16) and (17) are a suitable starting point to derive the asymptotic behaviour of the charge density and the current density for large x. In the following we will focus mainly on the current density. From (17) we see that for the latter we need to determine asymptotic expansions of the integrals

$$I_n(x) = \int_{k_n(\mu)}^\infty \mathrm{d}k \, [\mu - z_n(k)]^{1/2} (x-k) \times \left[\int_0^\infty \mathrm{d}t \, D^2_{z_n(k)-1/2}(\sqrt{2}(t-k)) \right]^{-1} D^2_{z_n(k)-1/2}(\sqrt{2}(x-k)). \tag{25}$$

It will turn out that $I_n(x)$ decays as $\exp(-x^2/2)$, so we can discard any terms that decay faster than that.

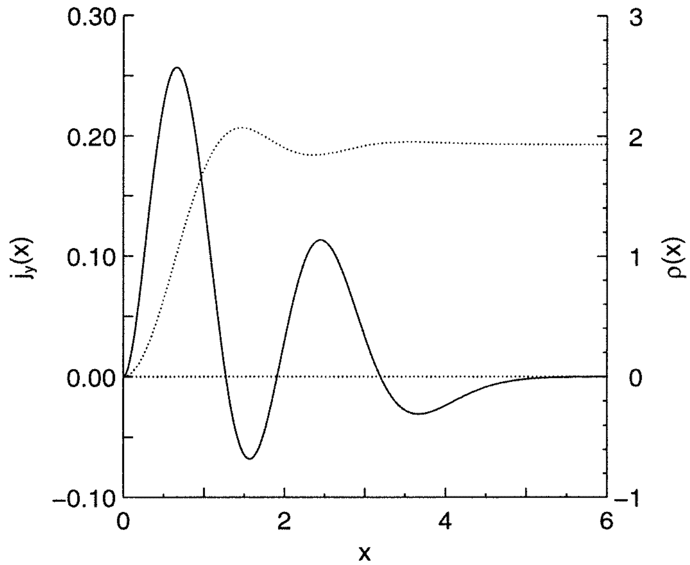

Figure 2. The charge density ($\cdots\cdots$, in units $e/2\pi^2(2m\omega_c/\hbar)^{3/2}$) and the current density (——, in units $e/2\pi^2(2m\omega_c/\hbar)^{3/2}(\hbar\omega_c/m)^{1/2}$) for $\mu = 2.0$.

We split the integration interval at $k' = \kappa x$, with $0 < \kappa < 1 - \frac{1}{2}\sqrt{2}$. The contribution to $I_n(x)$ from the interval $[k_n(\mu), k']$ can be estimated. Consider the normalization factor

$$\int_0^\infty \mathrm{d}t\, D_\nu^2(\sqrt{2}(t-k)) = \int_{-k}^\infty \mathrm{d}t\, D_\nu^2(\sqrt{2}t) \geqslant \int_{-k_n(\mu)}^\infty \mathrm{d}t\, D_\nu^2(\sqrt{2}t) \equiv c_n(\nu) \tag{26}$$

with $\nu = z_n(k) - \frac{1}{2}$, which implies that $\nu \in [n, \mu - \frac{1}{2}]$. Since $c_n(\nu)$ is finite in the closed interval $[n, \mu - \frac{1}{2}]$, we conclude that $c_n(\nu)$ is bounded from below by a certain c_n independent of k. Now we use the following asymptotic series [7]

$$D_\nu(z) \simeq \mathrm{e}^{-z^2/4} z^\nu A_\nu(z/\sqrt{2}) \tag{27}$$

which is valid for large and positive z. Here we introduced

$$A_\nu(z) = \sum_{m=0}^\infty \frac{(-\nu/2)_m((1-\nu)/2)_m}{m!}(-z^2)^{-m} \tag{28}$$

where $(a)_n$ is Pochhammer's symbol $a(a+1)\ldots(a+n-1)$. Note that $A_n(z)$ with n integer has a finite number of terms only; it is related to the Hermite polynomials by $H_n(z) = (2z)^n A_n(z)$. From $z_n(k) \leqslant \mu$ we conclude that

$$D^2_{z_n(k)-1/2}(\sqrt{2}(x-k)) \leqslant 2^{\mu-1/2}\mathrm{e}^{-(x-k)^2}(x-k)^{2\mu-1}[1 + \mathrm{O}((x-k)^{-2})] \tag{29}$$

for large positive $x - k$. This means that the contribution of the interval $[k_n(\mu), k']$ to $I_n(x)$ is smaller than

$$\frac{2^{\mu-1/2}\mu^{1/2}}{c_n}\int_{k_n(\mu)}^{k'} \mathrm{d}k\, \mathrm{e}^{-(x-k)^2}(x-k)^{2\mu}[1 + \mathrm{O}((x-k)^{-2})]. \tag{30}$$

Since we have chosen $k' < (1 - \frac{1}{2}\sqrt{2})x$, this decays faster than $\exp(-x^2/2)$, so it can be discarded.

For $k > k'$ we can use the asymptotic expansions of $z_n(k)$

$$[z_n(k) - (n + \tfrac{1}{2})] \simeq \frac{1}{\sqrt{\pi} n!} 2^n \mathrm{e}^{-k^2} k^{2n+1} \frac{A_n(k)}{B_n(k)} \tag{31}$$

(see appendix A) and of the normalization factor

$$\left[\int_0^\infty \mathrm{d}t\, D^2_{z_n(k)-1/2}(\sqrt{2}(t-k))\right]^{-1} \simeq \frac{1}{\sqrt{\pi} n!} - \frac{1}{\pi (n!)^2} 2^{n+1} \mathrm{e}^{-k^2} k^{2n+1} C_n(k) \tag{32}$$

(see appendix B), both of which are valid for large k. Since $[z_n(k) - (n + \frac{1}{2})]$ is small, we can write

$$D^2_{z_n(k)-1/2}(\sqrt{2}(x-k)) = D^2_n(\sqrt{2}(x-k)) + \frac{\partial}{\partial \nu} D^2_\nu(\sqrt{2}(x-k))\bigg|_{\nu=n} [z_n(k) - (n + \tfrac{1}{2})] + \text{h.o.t.} \tag{33}$$

With the help of these expressions we find

$$\begin{aligned} I_n(x) \simeq &\int_{k'}^\infty \mathrm{d}k\, [\mu - (n + \tfrac{1}{2})]^{1/2} \frac{1}{\sqrt{\pi} n!} (x-k) D^2_n(\sqrt{2}(x-k)) \\ &- \int_{k'}^\infty \mathrm{d}k\, \frac{1}{2} [\mu - (n + \tfrac{1}{2})]^{-1/2} \frac{1}{\pi (n!)^2} 2^n \mathrm{e}^{-k^2} k^{2n+1} \\ &\times \frac{A_n(k)}{B_n(k)} (x-k) D^2_n(\sqrt{2}(x-k)) - \int_{k'}^\infty \mathrm{d}k\, [\mu - (n + \tfrac{1}{2})]^{1/2} \\ &\times \frac{1}{\pi (n!)^2} 2^{n+1} \mathrm{e}^{-k^2} k^{2n+1} C_n(k) (x-k) D^2_n(\sqrt{2}(x-k)) \\ &+ \int_{k'}^\infty \mathrm{d}k\, [\mu - (n + \tfrac{1}{2})]^{1/2} \frac{1}{\pi (n!)^2} 2^n \mathrm{e}^{-k^2} k^{2n+1} \\ &\times \frac{A_n(k)}{B_n(k)} (x-k)\, \frac{\partial}{\partial \nu} D^2_\nu(\sqrt{2}(x-k))\bigg|_{\nu=n} + \text{h.o.t.} \end{aligned} \tag{34}$$

The first term can be discarded. This can be seen by writing D_n in terms of the Hermite polynomial H_n [7]

$$D_n(z) = 2^{-n/2} \mathrm{e}^{-z^2/4} H_n(z/\sqrt{2}). \tag{35}$$

As $H^2_n(z)$ is even in z, we have

$$\int_{k'}^\infty \mathrm{d}k\, (x-k) D^2_n(\sqrt{2}(x-k)) = 2^{-n} \int_{2x-k'}^\infty \mathrm{d}k\, \mathrm{e}^{-(x-k)^2} (x-k) H^2_n(x-k). \tag{36}$$

Since k' is less than $(1 - \frac{1}{2}\sqrt{2})x$, this decays faster than $\exp(-x^2/2)$.

In the remaining terms of (34) we split the integration interval once more, now at $k'' = \lambda x$, with $\lambda > \frac{1}{2}\sqrt{2}$. The contribution from $k > k''$ in the second and the third term is negligible. This can be shown in the same way as we did for the first term. For the fourth term we use the following integral representation of the parabolic cylinder function [7]

$$D_\nu(z) = \sqrt{\frac{2}{\pi}} \mathrm{e}^{z^2/4} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t^2/2} \cos(\nu\pi/2 - zt) t^\nu \tag{37}$$

to show that

$$\begin{aligned} \frac{\partial}{\partial \nu} D^2_\nu(\sqrt{2}(x-k))\bigg|_{\nu=n} = &\frac{2^{-n/2+3/2}}{\sqrt{\pi}} H_n(x-k) \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t^2/2} t^n \\ &\times \left\{\cos[n\pi/2 - \sqrt{2}(x-k)t] \ln t - \frac{\pi}{2} \sin[n\pi/2 - \sqrt{2}(x-k)t]\right\}. \end{aligned} \tag{38}$$

The absolute value of the part between curly brackets is smaller than $|\ln t| + \pi/2$, which implies that

$$\left| \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t^2/2} t^n \{\cos[n\pi/2 - \sqrt{2}(x-k)t] \ln t - \frac{\pi}{2} \sin[n\pi/2 - \sqrt{2}(x-k)t]\} \right| \leqslant c'_n \qquad (39)$$

where c'_n is independent of k and x. As a consequence we find that

$$\left| \int_{k''}^\infty \mathrm{d}k\, \mathrm{e}^{-k^2} k^{2n+1} \frac{A_n(k)}{B_n(k)} (x-k) \frac{\partial}{\partial \nu} D_\nu^2(\sqrt{2}(x-k)) \Big|_{\nu=n} \right|$$
$$\leqslant c''_n \int_{k''}^\infty \mathrm{d}k\, \mathrm{e}^{-k^2} k^{2n+1} |(x-k) H_n(x-k)| \qquad (40)$$

where c''_n is independent of k and x as well. The right-hand side decays faster than $\exp(-x^2/2)$ since we have taken $k'' > \frac{1}{2}\sqrt{2}x$.

We now collect all remaining terms, evaluating the quotient $A_n(k)/B_n(k)$ as a single series, writing D_n in terms of Hermite polynomials, and using

$$\frac{\partial}{\partial \nu} D_\nu(z) \Big|_{\nu=n} \simeq \mathrm{e}^{-z^2/4} z^n \left[A_n(z/\sqrt{2}) \ln z + \frac{\partial}{\partial \nu} A_\nu(z/\sqrt{2}) \Big|_{\nu=n} \right]. \qquad (41)$$

This asymptotic relation is valid for large and positive z and follows by differentiating (27) with respect to ν. The result for $I_n(x)$ as defined in (25) is

$$I_n(x) \simeq \frac{2^{2n+1}}{\pi (n!)^2} [\mu - (n + \tfrac{1}{2})]^{1/2} \int_{k'}^{k''} \mathrm{d}k\, \mathrm{e}^{-k^2} \mathrm{e}^{-(x-k)^2} k^{2n+1} (x-k)^{2n+1} P_n(k, x-k) \qquad (42)$$

with

$$P_n(k, x-k) = -\left\{ \frac{1}{4[\mu - (n+1/2)]} + \sum_{m=1}^n \frac{1}{m} - \gamma - \ln(2k(x-k)) \right\} K_n(k, x-k)$$
$$+ L_n(k, x-k) \qquad (43)$$

containing the asymptotic series

$$K_n(k, x-k) = 1 - \frac{1+n+n^2}{2} k^{-2} + \frac{n-n^2}{2} (x-k)^{-2} - \frac{4+9n-n^4}{8} k^{-4}$$
$$- \frac{n-n^4}{4} k^{-2} (x-k)^{-2} - \frac{3n - 6n^2 + 4n^3 - n^4}{8} (x-k)^{-4} + \cdots \qquad (44)$$

$$L_n(k, x-k) = -\frac{1+2n}{4} k^{-2} + \frac{1-2n}{4} (x-k)^{-2} - \frac{9-4n^3}{16} k^{-4}$$
$$- \frac{1-4n^3}{8} k^{-2} (x-k)^{-2} - \frac{3 - 12n + 12n^2 - 4n^3}{16} (x-k)^{-4} + \cdots. \qquad (45)$$

Integration over k gives us the following asymptotic expansion for $I_n(x)$

$$I_n(x) \simeq \frac{2^{-2n-3/2}}{\sqrt{\pi} (n!)^2} [\mu - (n + \tfrac{1}{2})]^{1/2} \mathrm{e}^{-x^2/2} x^{4n+2} R_n(x). \qquad (46)$$

Here we defined

$$R_n(x) = -\left\{ \frac{1}{4[\mu - (n+1/2)]} + \sum_{m=1}^n \frac{1}{m} - \gamma - \ln(x^2/2) \right\} M_n(x) + N_n(x) \qquad (47)$$

with the asymptotic series

$$M_n(x) = 1 - (3 + 2n + 4n^2) x^{-2} - (12 + 21n - 10n^2 - 8n^4) x^{-4} + \cdots \qquad (48)$$

$$N_n(x) = -(1+4n) x^{-2} - \frac{21 - 20n - 32n^3}{2} x^{-4} + \cdots. \qquad (49)$$

This result is independent of the particular choice of k' and k'' as it should be.

Finally, substitution of (46) into (17) yields the asymptotic expansion for the current density that we set out to establish. It has the form

$$j_y(x) \simeq -e\frac{m\omega_c^2}{2\pi^{5/2}\hbar}\sum_n{}' \frac{2^{-2n}}{(n!)^2}[\mu - (n+\tfrac{1}{2})]^{1/2}\mathrm{e}^{-x^2/2}x^{4n+2}R_n(x) \tag{50}$$

with the asymptotic series $R_n(x)$ as given in (47).

The asymptotic behaviour of the charge density can be determined in a similar fashion. One finds

$$\rho(x) - \rho \simeq e\frac{m^{3/2}\omega_c{}^{3/2}}{\pi^{5/2}\hbar^{3/2}}\sum_n{}' \frac{2^{-2n}}{(n!)^2}[\mu - (n+\tfrac{1}{2})]^{1/2}\mathrm{e}^{-x^2/2}x^{4n+1}R'_n(x) \tag{51}$$

where we introduced the abbreviation

$$R'_n(x) = -\left\{\frac{1}{4[\mu-(n+\frac{1}{2})]} + \sum_{m=1}^{n}\frac{1}{m} - \gamma - \ln(x^2/2)\right\}M'_n(x) + N'_n(x) \tag{52}$$

with the asymptotic series

$$M'_n(x) = 1 - (2+2n+4n^2)x^{-2} - (10+19n-6n^2-8n^4)x^{-4} + \cdots \tag{53}$$

$$N'_n(x) = -(1+4n)x^{-2} - \frac{19-12n-32n^3}{2}x^{-4} + \cdots \tag{54}$$

5. Discussion

To check the validity of our asymptotic expansions we compared them with numerical results for the charge and the current density. In figure 3 we plotted $I_0(x)$ for $\mu = 1.0$. For this value of μ there is only one (partially) filled Landau level, so $I_0(x)$ represents the complete current density. Because of its fast decay the prefactor $\exp(-x^2/2)x^{4n+2}$ has been divided out. The full curve corresponds to the numerical results, the dotted line to the asymptotic expansion (46). As can be seen, the convergence is quite good.

As (50) and (51) show, the contribution of each Landau level n to both the current density and the charge density has a Gaussian decay for large x (in leading order proportional to $\exp(-x^2/2)x^{4n+2}\ln(x^2/2)$ and $\exp(-x^2/2)x^{4n+1}\ln(x^2/2)$, respectively). Therefore, when only a limited number of Landau levels is filled, in other words for every finite magnetic field, both densities decay with a tail proportional to a Gaussian. This property has been established before in a preliminary report by one of us [11]. The purely algebraic prefactors (without a logarithmic dependence) reported there, were inferred from numerical evidence only and are not corroborated by the asymptotic series presented above.

The decay found here is consistent with the bound derived by Macris *et al* [6]. However, it disagrees with the results of Ohtaka and Moriya [2] and of Jancovici [3]. In the latter paper the current density at $T = 0$ is given as

$$j_y(x) = e\frac{m\mu\omega_c^2}{16\pi^2\hbar}\left\{8\mu x^2\left[\frac{\pi}{2} - \mathrm{Si}(2^{3/2}\mu^{1/2}x)\right] + \left(\frac{3}{4\mu x^2} - 1\right)\sin(2^{3/2}\mu^{1/2}x) - \left(\frac{3}{2^{1/2}\mu^{1/2}x} + 2^{3/2}\mu^{1/2}x\right)\cos(2^{3/2}\mu^{1/2}x)\right\} \tag{55}$$

with $\mathrm{Si}(z)$ the sine integral. The right-hand side decays algebraically, with a tail proportional to x^{-1} for large x. It is obtained via an inverse Laplace transform of the current density

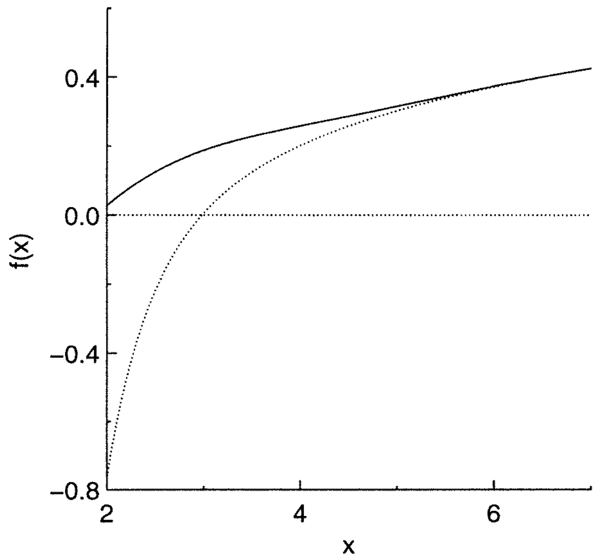

Figure 3. Comparison between numerical results (——) and the asymptotic expansion (······) of the current density for $\mu = 1.0$. The plotted function is $f(x) = \exp(x^2/2)x^{-2}I_0(x)$, with $I_0(x)$ as defined in (25).

$j_y^{\mathrm{MB}}(\beta, x)$ for a magnetized free-electron gas with Maxwell–Boltzmann statistics [12]:

$$j_y(x) = \frac{1}{\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\beta}{\beta} \frac{Z}{N} j_y^{\mathrm{MB}}(\beta, x) \mathrm{e}^{\beta\mu} \tag{56}$$

where Z is the one-particle partition function for Maxwell–Boltzmann statistics and N is the number of particles. The Maxwell–Boltzmann form of the current density employed in [3] is obtained by a linear-response method valid for small magnetic field. In fact, the dimensionless parameter that has to be small is $\beta\hbar\omega_c$. The integration in (56) is taken over all values of β, and thus in particular over all values of $\beta\hbar\omega_c$. Hence, it is not justified *a priori* to insert the linear-response expression for $j_y^{\mathrm{MB}}(\beta, x)$ and to carry out the integration subsequently. As a consequence, expression (55), and the ensuing algebraic decay is not guaranteed to be correct. As has been already remarked in the introduction, the procedure of taking inverse Laplace transforms of Maxwell–Boltzmann expressions for small fields may even lead to weird effects like undamped oscillations, if it is applied to other physical quantities. Questions about the validity of (55) in the limit $x \to \infty$ have been raised before by Shishido [13], who argues that the expression is not uniformly convergent, and is valid only for small x (and small B).

It should be noted here that our asymptotic expansions (50) and (51) are rather awkward when it comes to studying the limit $B \to 0$. In that limit the number of filled Landau levels goes to infinity. The coefficients in the expansion rapidly grow with the label n of the Landau level, as is clear from (48), (49), (53) and (54). Hence, the asymptotic region moves further and further away from the wall, as B goes to 0.

Our approach to determine the asymptotic behaviour of profiles for finite magnetic fields can easily be generalized to other physical quantities, for instance the kinetic pressure. In general, the leading term is proportional to $\exp(-x^2/2)x^m \ln(x^2/2)$, where m increases with the number of filled Landau levels and with the number of particle momenta occurring as factors in the expression for the physical quantity being calculated.

Acknowledgments

This investigation is part of the research programme of the 'Stichting voor Fundamenteel Onderzoek der Materie (FOM)', which is financially supported by the 'Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)'.

Appendix A. Asymptotics of $z_n(k)$

In section 3 we introduced the function $z_n(k)$, which defines the eigenvalues of the Fourier-transformed Hamiltonian (12). It is defined by

$$D_{z_n(k)-1/2}(-\sqrt{2}k) = 0. \tag{A1}$$

The asymptotic expansion of $D_\nu(-\sqrt{2}k)$ for large and positive k is given by [7]

$$D_\nu(-\sqrt{2}k) \simeq \mathrm{e}^{\mathrm{i}\pi\nu}\mathrm{e}^{-k^2/2}(\sqrt{2}k)^\nu A_\nu(k) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)}\mathrm{e}^{k^2/2}(\sqrt{2}k)^{-\nu-1}B_\nu(k) \tag{A2}$$

with A_ν as defined in (28) and with B_ν given by

$$B_\nu(k) = \sum_{m=0}^{\infty}\frac{((1+\nu)/2)_m((2+\nu)/2)_m}{m!}(k^2)^{-m}. \tag{A3}$$

Setting (A2) to zero and expanding around $\nu = n$ we arrive at

$$[z_n(k) - (n+\tfrac{1}{2})] \simeq \frac{1}{\sqrt{\pi}n!}2^n\mathrm{e}^{-k^2}k^{2n+1}\frac{A_n(k)}{B_n(k)} \tag{A4}$$

for large positive k. This is a generalization of the expression given by Kunz [9].

Appendix B. Asymptotics of the normalization factor

In section 4 we needed the asymptotic expansion of the normalization factor $[\int_0^\infty \mathrm{d}t\, D^2_{z_n(k)-1/2}(\sqrt{2}(t-k))]^{-1}$ for large k. In appendix A we have seen that for large k the function $[z_n(k) - (n+\frac{1}{2})]$ is small. Therefore we can write

$$\begin{aligned}\int_0^\infty \mathrm{d}t\, D^2_{z_n(k)-1/2}(\sqrt{2}(t-k)) &= \int_0^\infty \mathrm{d}t\, D^2_n(\sqrt{2}(t-k)) \\ &\quad + \frac{\partial}{\partial\nu}\int_0^\infty \mathrm{d}t\, D^2_\nu(\sqrt{2}(t-k))\bigg|_{\nu=n}[z_n(k)-(n+\tfrac{1}{2})] \\ &\quad + \frac{\partial^2}{\partial\nu^2}\int_0^\infty \mathrm{d}t\, D^2_\nu(\sqrt{2}(t-k))\bigg|_{\nu=n}\tfrac{1}{2}[z_n(k)-(n+\tfrac{1}{2})]^2 + \text{h.o.t.}\end{aligned} \tag{B1}$$

The first term in this expansion is given by

$$\begin{aligned}\int_0^\infty \mathrm{d}t\, D^2_n(\sqrt{2}(t-k)) &= \int_{-\infty}^\infty \mathrm{d}t\, D^2_n(\sqrt{2}t) - \int_{-\infty}^{-k}\mathrm{d}t\, D^2_n(\sqrt{2}t) \\ &\simeq \sqrt{\pi}n! - 2^{n-1}\mathrm{e}^{-k^2}k^{2n+1}\left(k^{-2} - \frac{1-3n+n^2}{2}k^{-4} + \cdots\right)\end{aligned} \tag{B2}$$

as can be derived by expressing D_n in terms of the Hermite polynomial H_n, followed by term by term integration of the resulting series.

With the help of the integral representation (37) of D_ν the coefficient of the second term in (B1) becomes

$$\frac{\partial}{\partial\nu}\int_0^\infty \mathrm{d}t\, D_\nu^2(\sqrt{2}(t-k))\bigg|_{\nu=n} = \frac{2\sqrt{2}}{\sqrt{\pi}}\int_{-k}^\infty \mathrm{d}s\,\mathrm{e}^{s^2/2}D_n(\sqrt{2}s) \times\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t^2/2}t^n\left[\cos(n\pi/2-\sqrt{2}st)\ln t - \frac{\pi}{2}\sin(n\pi/2-\sqrt{2}st)\right]. \tag{B3}$$

Repeated partial integration yields

$$\frac{\partial}{\partial\nu}\int_0^\infty \mathrm{d}t\, D_\nu^2(\sqrt{2}(t-k))\bigg|_{\nu=n} = \frac{1}{\sqrt{\pi}}\sum_{m=0}^{n}(-1)^m 2^{-n/2+m/2+1}\frac{n!}{(n-m)!}H_{n-m}(s) \times\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t^2/2}t^{n-m-1}\Big\{\cos[(n+m+1)\pi/2-\sqrt{2}st]\ln t - \frac{\pi}{2}\sin[(n+m+1)\pi/2-\sqrt{2}st]\Big\}\bigg|_{s=-k}^{\infty}. \tag{B4}$$

For all $m<n$ we write the contribution at $s=-k$ of the sine term as

$$-\frac{\pi}{2}\,\mathrm{Im}\left[\mathrm{i}^{n+m+1}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t^2/2}t^{n-m-1}\mathrm{e}^{\mathrm{i}\sqrt{2}kt}\right]. \tag{B5}$$

We now use a theorem [14] stating that for large x the Fourier integral

$$\int_\alpha^\beta \mathrm{d}t\,\phi(t)\mathrm{e}^{\mathrm{i}xt} \tag{B6}$$

has an asymptotic expansion to which the endpoint α contributes as

$$A = \sum_{n=0}^\infty \mathrm{i}^{n+1}\frac{\mathrm{d}^n\phi(\alpha)}{\mathrm{d}\alpha^n}x^{-n-1}\mathrm{e}^{\mathrm{i}x\alpha} \tag{B7}$$

if $\phi(t)$ has no singularity in $[\alpha,\beta]$. With the help of this theorem we can show that

$$-\frac{\pi}{2}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t^2/2}t^{n-m-1}\sin[(n+m+1)\pi/2+\sqrt{2}kt] \simeq -\frac{\pi}{2}(-1)^n(\sqrt{2}k)^{-n+m}\sum_{l=0}^\infty\frac{(2l+n-m-1)!}{2^{2l}l!}k^{-2l}. \tag{B8}$$

The contribution of the cosine term at $s=-k$ in (B4) can be written as

$$\mathrm{Re}\left[\mathrm{i}^{n+m+1}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t^2/2}t^{n-m-1}\mathrm{e}^{\mathrm{i}\sqrt{2}kt}\ln t\right]. \tag{B9}$$

Because of the logarithm we need a generalization of the previous theorem to Fourier integrals with logarithmic singularities. This generalization can also be found in [14]. It states that for $\phi(t)=\phi_1(t)\ln(t-\alpha)$ the asymptotic expansion of (B6) contains a contribution from the lower endpoint which reads

$$A = \sum_{n=0}^\infty \mathrm{i}^{n+1}\frac{\mathrm{d}^n\phi_1(\alpha)}{\mathrm{d}\alpha^n}\left[\psi(n+1)-\ln x+\mathrm{i}\frac{\pi}{2}\right]x^{-n-1}\mathrm{e}^{\mathrm{i}x\alpha} \tag{B10}$$

with $\psi(z)$ the logarithmic derivative of the gamma function. With the help of this theorem we see that the contribution from $s=-k$ of the cosine term is identical to the contribution of the sine term. Using the same method, we find that for $s\to\infty$ the two terms in (B4) cancel, at least for $m<n$.

The contributions for $m = n$ can be calculated in a similar fashion, although they need some extra attention because of the additional t^{-1} singularity. They add up to

$$\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t^2/2} t^{-1} \{\cos[(2n+1)\pi/2 - \sqrt{2}st] \ln t - \frac{\pi}{2} \sin[(2n+1)\pi/2 - \sqrt{2}st]\}\Big|_{s=-k}^{\infty}$$
$$\simeq \pi(-1)^n \left[-\gamma - \ln(\sqrt{2}k) + \sum_{l=1}^{\infty} \frac{(2l-1)!}{2^{2l} l!} k^{-2l} \right] \qquad \text{(B11)}$$

where γ is Euler's constant. Collecting all these terms we obtain

$$\frac{\partial}{\partial \nu} \int_0^\infty \mathrm{d}t\, D_\nu^2(\sqrt{2}(t-k))\Big|_{\nu=n} \simeq 2\sqrt{\pi} n! \left[\sum_{m=1}^{n} \frac{1}{m} - \gamma - \ln(\sqrt{2}k) \right.$$
$$\left. + \frac{1+2n}{4} k^{-2} + \frac{3+6n+6n^2}{16} k^{-4} + \cdots \right]. \qquad \text{(B12)}$$

Finally, we have for the third term in (B1)

$$\frac{1}{2} \frac{\partial^2}{\partial \nu^2} \int_0^\infty \mathrm{d}t\, D_\nu^2(\sqrt{2}(t-k))\Big|_{\nu=n} = \int_0^\infty \mathrm{d}t\, D_\nu(\sqrt{2}(t-k))\, \frac{\partial^2}{\partial \nu^2} D_\nu(\sqrt{2}(t-k))\Big|_{\nu=n}$$
$$+ \int_0^\infty \mathrm{d}t\, \left[\frac{\partial}{\partial \nu} D_\nu(\sqrt{2}(t-k)) \right]_{\nu=n}^2 . \qquad \text{(B13)}$$

An asymptotic expansion of the first integral can be derived along the same route as above. The only new ingredient is a straightforward extension of the theorem by Erdélyi to Fourier integrals with squared logarithmic singularities. It states that if in (B6) one takes $\phi(t) = \phi_2(t) \ln^2(t-\alpha)$, the contribution of the lower boundary is given by the asymptotic expansion

$$A = \sum_{n=0}^{\infty} \mathrm{i}^{n+1} \frac{\mathrm{d}^n \phi_2(\alpha)}{\mathrm{d}\alpha^n} \left\{ [\psi(n+1) - \ln x]^2 + \zeta(2,n) + \mathrm{i}\pi[\psi(n+1) - \ln x] - \frac{\pi^2}{4} \right\}$$
$$\times x^{-n-1} \mathrm{e}^{\mathrm{i}x\alpha} \qquad \text{(B14)}$$

with $\zeta(k,n)$ the generalized zeta function. As a consequence the asymptotics of the first integral in (B13) is found to be of order $\ln^2(\sqrt{2}k)$. This implies that for large k the first integral is negligible with respect to the second, as we shall see.

The asymptotic behaviour of the second integral in (B13) follows by noting that for large k the dominant contribution comes from the lower end of the integration interval. With the help of (37), (B7) and (B10) we can derive the following asymptotic expansion for the integrand

$$\frac{\partial}{\partial \nu} D_\nu(z)\Big|_{\nu=n} \simeq \sqrt{2\pi} n! \mathrm{e}^{z^2/4} z^{-n-1} B_n(z/\sqrt{2}) \qquad \text{(B15)}$$

for large and negative z. Term-by-term integration leads to

$$\frac{1}{2} \frac{\partial^2}{\partial \nu^2} \int_0^\infty \mathrm{d}t\, D_\nu^2(\sqrt{2}(t-k))\Big|_{\nu=n}$$
$$\simeq \pi(n!)^2 2^{-n-1} \mathrm{e}^{k^2} k^{-2n-1} \left(k^{-2} + \frac{5+5n+n^2}{2} k^{-4} + \cdots \right). \qquad \text{(B16)}$$

The right-hand side grows exponentially as $k \to \infty$, but this is compensated by the factor $\exp(-2k^2)$ in $[z_n(k) - (n + \frac{1}{2})]^2$, resulting in an overall $\exp(-k^2)$ behaviour. That is the reason why we had to expand (B1) up to second order in $[z_n(k) - (n + \frac{1}{2})]$. Higher-order derivatives of $\int_0^\infty \mathrm{d}t\, D_\nu^2(\sqrt{2}(t-k))$ are also of order $\exp(k^2)$ or less, so that we do not have to go beyond second order.

Substitution of (A4), (B2), (B12) and (B16) in (B1) gives

$$\left[\int_0^\infty \mathrm{d}t\, D^2_{z_n(k)-1/2}(\sqrt{2}(t-k))\right]^{-1} \simeq \frac{1}{\sqrt{\pi}n!} - \frac{1}{\pi(n!)^2}2^{n+1}\mathrm{e}^{-k^2}k^{2n+1}C_n(k) \tag{B17}$$

with the asymptotic series

$$C_n(k) = \left[\sum_{m=1}^{n}\frac{1}{m} - \gamma - \ln(\sqrt{2}k)\right]\left(1 - \frac{1+n+n^2}{2}k^{-2} - \frac{4+9n-n^4}{8}k^{-4} + \cdots\right) + \left(\frac{1+2n}{4}k^{-2} + \frac{9-4n^3}{16}k^{-4} + \cdots\right). \tag{B18}$$

References

[1] Landau L 1930 *Z. Phys.* **64** 629

[2] Ohtaka K and Moriya T 1973 *J. Phys. Soc. Japan* **34** 1203

[3] Jancovici B 1980 *Physica* **101A** 324

[4] John P and Suttorp L G 1995 *J. Phys. A: Math. Gen.* **28** 6087

[5] John P and Suttorp L G 1994 *Physica* **210A** 237

[6] Macris N, Martin Ph A and Pulé J V 1988 *Commun. Math. Phys.* **117** 215

[7] Magnus W, Oberhettinger F and Soni R P 1966 *Formulas and Theorems for the Special Functions of Mathematical Physics* (Berlin: Springer)

[8] MacDonald A H and Středa P 1984 *Phys. Rev.* B **29** 1616

[9] Kunz H 1994 *J. Stat. Phys.* **76** 183

[10] Fal'kovskiĭ L A 1970 *Sov. Phys.–JETP* **31** 981

[11] Suttorp L G 1996 *Proc. Int. Conf. on Physics of Strongly Coupled Plasmas* ed W D Kraeft and M Schlanges (Singapore: World Scientific) p 381

[12] Sondheimer E H and Wilson A H 1951 *Proc. R. Soc.* A **210** 173

[13] Shishido F 1991 *Phys. Lett.* A **152** 437

[14] Erdélyi A 1956 *J. Soc. Indust. Appl. Math.* **4** 38